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Statistical properties of spatiotemporal dynamical systems

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In this paper we consider spatiotemporal dynamical systems modeled by coupled, or uncoupled but noise-driven, map lattices. In particular, we examine reports in the literature where it is found that the distribution of certain mean-field quantities violates the law of large numbers (hence nonstatistical) but not the central-limit theorem. Our results show that the origin of such nonstatistical behavior is due to the statistical dependence between random variables at different lattice sites, thus rendering nonapplicable to such situations the law of large numbers and the central-limit theorem. Additional issues explored include the discussion of a special class of systems where nonstatistical behavior is not observed and the physical motivation for considering uncoupled but noise-driven map lattices.

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Recently, it has been shown (see [1] and references therein) that certain aspects of chaotic spatiotemporal dynamical systems such as a turbulent fluid can be modeled by coupled map lattices (CML's) of the following type:

$$x_{n+1}^i = F^i(\{x_n^j\}_{j=1}^L, \epsilon) , \tag{1}$$

where x_n^i denotes the value of the continuous variable x^i at discrete time n, with $i = 1, 2, \ldots, L$ labeling the site in the lattice, and ϵ is used here to give a general measure of the coupling strength. If we take the coupling to be mean field, we are led to the study of a special class of CML, an example of which is

$$x_{n+1}^i = (1-\epsilon) f(x_n^i) + \frac{\epsilon}{L} \sum_{j=1}^{L} f(x_n^j) , \tag{2}$$

with $f(x)$ prescribing the local dynamics at each lattice site. We assume in the absence of coupling ($\epsilon = 0$) that each individual map exhibits chaotic dynamics.

A different but related class of spatiotemporal model systems being considered [2] is the following noise-driven uncoupled map lattice (UCML):

$$x_{n+1}^i = F(x_n^i, a_n^i) , \tag{3}$$

where a_n^i can be a random variable influencing the dynamics at site i and $F(x_n^i, 0) = f(x_n^i)$. This type of model may be motivated in part by considering a hypothetical physical situation in which a system consisting of L identical chaotic units is embedded in a noisy environment. If the coupling between units is sufficiently weak, it can be neglected in the first approximation. There are four different cases regarding the nature of the environment. In case (1), $a_n^i = a$, meaning that the background is homogeneous and constant. Case (2) concerns the possibility that the environment varies randomly in time but is correlated in space. That is, $a_n^i = \xi_n^i$, where ξ_n^i is a random variable of zero mean with $\langle \xi_n^i \xi_n^j \rangle \neq 0$. The symbol $\langle\ \rangle$ is used to denote ensemble average. Specifically, one may consider $a_n^i = \xi_n$, corresponding to a random background that is homogeneous in space. In this paper we suppose that the random variables representing the background at different lattice sites are all uncorrelated in time (white noise). The third case differs from the second by having $a_n^i = \xi^i$. Namely, the background is nonhomogeneous in space but remains a constant of time. The evolution of different units in this case follows different laws but is deterministic once the variable ξ^i is assigned a specific value. Case (4) arises when the environment is random in time and uncorrelated in space. That is, $a_n^i = \xi_n^i$, where $\langle \xi_n^i \xi_m^j \rangle = 0$ for $n \neq m$ and $i \neq j$. Clearly, case (2) depicts the situation which is most likely to occur in practice. In particular, the specific example of $a_n^i = \xi_n$ can be regarded as a model of Eq. (2), provided the dynamics of the latter is sufficiently chaotic. For simplicity, we may henceforth use case (i), $i = 1,2,3,4$, to refer to the four different situations associated with Eq. (3).

From the viewpoint of physical measurement, a quantity of interest for both Eqs. (2) and (3) is the distribution of some mean-field variables, corresponding possibly to observable quantities such as

$$h = \frac{1}{L}\sum_{i=1}^{L} f(x^i) , \tag{4}$$

and their statistical characteristics (e.g., mean and variance), where $f(x)$ is the function defining the local dynamics. It is reported that the histogram constructed from the temporal realizations of h, denoted h_n, appears to be normally distributed, but for Eq. (2) and case (2) of Eq. (3), the variance of h,

$$\Sigma^2 = \langle (h - \langle h \rangle)^2 \rangle , \tag{5}$$

exhibits "nonstatistical" behavior [2–8]. Here $\langle h \rangle$ is the mean of h. Specifically, if $\ln\Sigma^2$ is plotted against $\ln L$, as schematically illustrated in Fig. 1, one observes a straight line with a slope of -1 for cases (1), (3), and (4) (statistical), but a curve approaching a constant for large enough L for case (2) and Eq. (2) (nonstatistical). The purpose of this paper is to present results addressing the origin of this observation, as well as discuss other related issues.

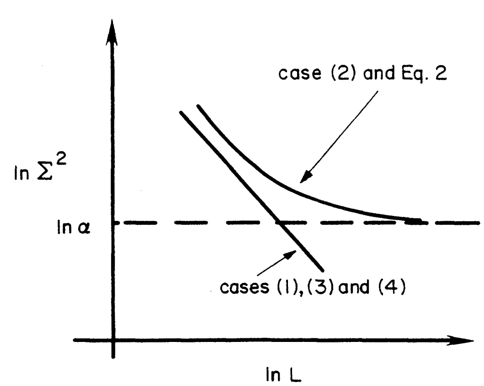

FIG. 1. Schematic illustration of $\ln\Sigma^2$ as a function of $\ln L$ for four different cases of Eq. (3) and for Eq. (2).

The mean value of h is defined as

$$\langle h \rangle = \frac{1}{L}\sum_{i=1}^{L} \langle f(x^i) \rangle . \tag{6}$$

In actual calculations the ensemble average (or the average over invariant measures for deterministic variables) may be replaced by the time average. From Eq. (5), and using Eqs. (4) and (6), we have

$$\begin{aligned}\Sigma^2 &= \left\langle \left\{ \frac{1}{L}\sum_{i=1}^{L} [f(x^i) - \langle f(x^i) \rangle] \right\}^2 \right\rangle \\ &= \frac{1}{L^2}\sum_{i=1}^{L} \langle [f(x^i) - \langle f(x^i) \rangle]^2 \rangle \\ &\quad + \frac{1}{L^2}\sum_{\substack{i,j \\ i\neq j}} \langle [f(x^i) - \langle f(x^i) \rangle][f(x^j) - \langle f(x^j) \rangle] \rangle \\ &= \frac{1}{L^2}\sum_{i=1}^{L} \sigma_i^2 + \frac{1}{L^2}\sum_{\substack{i,j \\ i\neq j}} \alpha_{ij} ,\end{aligned}$$

where σ_i^2 is the variance of the random variable $f(x^i)$, and α_{ij} is the covariance between $f(x^i)$ and $f(x^j)$. Here we use the term random variable to refer to the dynamical variable at a given lattice site regardless of whether the local dynamics is deterministic or random. For cases (1), (2), and (4) of Eq. (3) and Eq. (2), since $\sigma_i^2 = \sigma^2$ and $\alpha_{ij} = \alpha$ are independent of the lattice sites, we obtain

$$\begin{aligned}\Sigma^2 &= \frac{\sigma^2}{L} + \frac{L(L-1)}{L^2}\alpha , \\ &\approx \frac{\sigma^2}{L} + \alpha .\end{aligned} \tag{7}$$

The second step is warranted for large L. For case (3), since every unit evolves differently, σ_i^2 is generally site dependent. When L is large, however, we can use the approximation $\sum_{i=1}^{L}\sigma_i^2 \approx L\bar{\sigma}^2$, with $\bar{\sigma}^2$ the mean value of σ_i^2. In what follows, we show that, for this case as well as for cases (1) and (4), the covariance between random variables at different sites is zero, resulting from their statistical independence, thus explaining the observed linear behavior in Fig. 1. We further show that, except for a special class of systems, which includes the much-studied tent map, the covariance α is generally not zero for case (2) and for Eq. (2), and is the observed asymptotic value approached by Σ^2 in the limit $L \to \infty$.

We say two random variables are statistically independent if their joint probability distribution function is the product of the two individual distribution functions, which, in the case of deterministic variables [cases (1) and (3) of Eqs. (3) and (2)], refer to the invariant measure on the attractor. It suffices to consider the situation of two uncoupled random maps expressed as

$$u_{n+1} = G(u_n, \xi_n) , \tag{8}$$

$$v_{n+1} = H(v_n, \eta_n) , \tag{9}$$

with ξ_n and η_n two random variables. Here we allow the two maps to be different. The joint probability distribution function of u and v at time n, $P_n(u,v)$, is related to that at time $n-1$ via

$$P_n(u,v)=\int \delta(u-G(u',\xi))\delta(v-H(v',\eta))P_{n-1}(u',v') \times W_{\xi\eta}(\xi,\eta)du'dv'd\xi\,d\eta \ ,$$

where $W_{\xi\eta}(\xi,\eta)$ is the joint probability distribution function of ξ and η, and $\delta(\ )$ denotes the standard Dirac δ function. In the limit $n\rightarrow\infty$ one obtains the stationary distribution $P(u,v)$ as the fixed point of the above functional iteration

$$P(u,v)=\int \delta(u-G(u',\xi))\delta(v-H(v',\eta))P(u',v') \times W_{\xi\eta}(\xi,\eta)du'dv'd\xi\,d\eta \ . \tag{10}$$

If the random variables ξ and η are statistically independent, then their joint distribution can be written as $W_{\xi\eta}(\xi,\eta)=W_\xi(\xi)W_\eta(\eta)$. Note that, in addition to case (4), cases (1) and (3) are also included in this category by writing $W_\xi(\xi)$ and $W_\eta(\eta)$ as δ functions. Since the maps are uncoupled, a solution of the form $P(u,v)=P_u(u)P_v(v)$ satisfies Eq. (10), where $P_u(u)$ and $P_v(v)$ are stationary distributions for u and v, respectively. Assuming that for Eq. (10) there exists a unique solution, the covariance between functions of u and that of v is zero for all three cases. This leads to $\alpha=0$ in Eq. (7), and hence the observed linear behavior in Fig. 1.

If ξ and η are not independent random variables, that is, $\langle\xi\eta\rangle\neq 0$, their joint probability distribution function cannot be written as the product of the two individual distributions. Consequently, a solution of the type $P(u,v)=P_u(u)P_v(v)$ generally does not satisfy Eq. (10), resulting in finite covariance between variables which are functions of u and variables which are functions of v. This is what underlies the observed saturation of Σ^2 for case (2) and for Eq. (2) for the examples considered in the literature. The saturated value of Σ^2 for case (2) and for Eq. (2) is α from Eq. (7). We note that in these cases, since the sampling at site i depends on that at site j, both the central-limit theorem and the law of large numbers do not apply. An interesting physical implication of the above results is that one can learn about the correlation properties between the subunits in the system by examining the behavior of the variance of certain mean-field variables as a function of the system size.

It is found that for systems such as the tent map one observes linear behavior in the plots of $\ln\Sigma^2$ versus $\ln L$ for Eq. (2) [8] and for all the cases of Eq. (3). We explain the origin of this phenomenon below. We also point out that the same phenomenon should occur for a wider class of systems.

For concreteness, consider an example map defined on the unit interval shown in Fig. 2. The slopes in this case satisfy the relationship:

$$1/\beta_1+1/\beta_2=1 \ . \tag{11}$$

Assuming the noise is additive, i.e.,

$$u_{n+1}=g(u_n)+\xi_n \bmod 1 \ =G(u_n,\xi_n) \ ,$$

where $g(u)$ is the function shown in the figure, by identi-

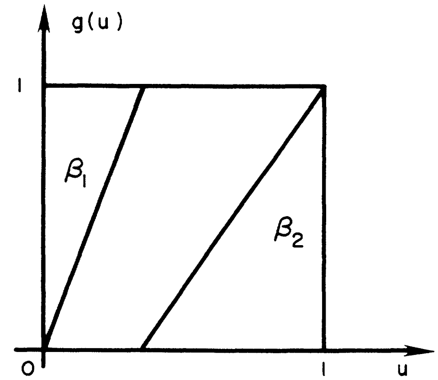

FIG. 2. A piecewise stretching map defined on the unit circle.

fying 1 with 0 (mod 1), the random dynamics can be regarded as taking place on a circle of unit length. The stationary distribution function for u obeys

$$P_u(u)=\int \delta(u-G(u',\xi))P_u(u')W_\xi(\xi)du'd\xi \ . \tag{12}$$

The integration over u' can be carried out, and the result is

$$P_u(u)=\int \frac{P_u(u_1)}{\beta_1}W_\xi(\xi)d\xi+\int \frac{P_u(u_2)}{\beta_2}W_\xi(\xi)d\xi \ ,$$

where u_1 and u_2 denote the two solutions of u' to the equation $u=g(u')+\xi \bmod 1$. It is clear that $P_u(u)=1$, namely the uniform distribution, is a solution to the above equation, by virtue of Eq. (11). Again we assume that this is the only solution to Eq. (12). This assumption is reasonable, given the uniform-expanding nature of the map, and is supported by numerical simulations.

The stationary joint probability distribution function of u and v for the following pair of maps,

$$u_{n+1}=g(u_n)+\xi_n \bmod 1 \ =G(u_n,\xi_n) \ , \tag{13}$$

$$v_{n+1}=g(v_n)+\eta_n \bmod 1 \ =G(v_n,\eta_n) \ , \tag{14}$$

satisfies

$$P(u,v)=\int \delta(u-G(u',\xi))\delta(v-G(v',\eta))P(u',v') \times W_{\xi\eta}(\xi,\eta)du'dv'd\xi\,d\eta \ .$$

Performing the integration over u' and v' we obtain

$$P(u,v)=\int \frac{P(u_1,v_1)}{\beta_1\beta_1}W_{\xi\eta}(\xi,\eta)d\xi\,d\eta + \int \frac{P(u_1,v_2)}{\beta_1\beta_2}W_{\xi\eta}(\xi,\eta)d\xi\,d\eta + \int \frac{P(u_2,v_1)}{\beta_2\beta_1}W_{\xi\eta}(\xi,\eta)d\xi\,d\eta + \int \frac{P(u_2,v_2)}{\beta_2\beta_2}W_{\xi\eta}(\xi,\eta)d\xi\,d\eta \ ,$$

with u_1,u_2 and v_1,v_2 solutions of u' and v' to the equations $u=g(u')+\xi \bmod 1$ and $v=g(v')+\eta \bmod 1$, respectively. Evidently, $P(u,v)=1$ is a solution to the above integral equation. Since this function $P(u,v)=1$ can be viewed as the product of $P_u(u)=1$ and $P_v(v)=1$, the variables u and v are statistically independent, regardless of whether ξ and η are independent or not. The same argument and conclusion can be applied to maps defined by arbitrary piecewise linear functions on circles, provided the slope of each piece β_k satisfies

$$\sum_{k=1}^{K} \frac{1}{|\beta_k|}=1 ,$$

where K is the total number of pieces of the linear function. This general class of maps includes the tent map as a special case. If the noise is parametric, however, for the same class of maps, the covariance between random variables at different sites is usually finite, if the noise is correlated in space. We have confirmed this in our numerical experiments.

As mentioned earlier, case (2) of Eq. (3) can be thought of as a model for Eq. (2), if the latter is sufficiently chaotic. Discussions above support this hypothesis insofar as the statistical properties of certain mean-field variables are concerned. On the other hand, regarding other aspects of the same variables, these two classes of situations may differ drastically from one another, giving rise to diagnostic signatures in experiments. For instance, it is reported [5,6] that for the case of Eq. (2), the saturated value α of Σ^2 scales with the strength of coupling ϵ as

$$\alpha \sim \epsilon^{2M} ,$$

where M is the dimension of the local dynamics ($M=1$ for the models we consider in this paper). No such scaling is observed for case (2) of Eq. (3). Furthermore, the time series $\{h_n\}$ is shown to exhibit interesting coherence behavior characterized by peaks in its power spectrum for the case of Eq. (2). Again no such coherence is apparent in the spectrum of a similar time series generated by case (2) of Eq. (3). It is further revealed [8] that the spectral peaks are related to the circular motion of the data points reconstructed in a two-dimensional phase space using delay coordinates. The origin of this coherent behavior remains to be understood.

In summary, we stress that spatially correlated random variables can induce correlation among random dynamical variables at different lattice sites. This underlies the nonapplicability to such situations of the law of large numbers and the central-limit theorem. The covariance between random variables at two different sites gives rise to the observed saturation value approached by the function of $\ln\Sigma^2$ versus $\ln L$.

Note added in proof. Recently, the following relevant comment appeared: A. S. Pikovsky, Phys. Rev. Lett. **71**, 653 (1993).

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